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Algorithms for some geometric properties of tangential intersection of two surfaces in Euclidean 3-spaces.

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Abstract

In this paper, we study the geodesic curvature of intersection curve of two regular parametric and two regular implicit surfaces in $\mathbb{R}^3$. Here the intersection curve will be of tangential type, i.e., the normal vectors of the two surfaces at the given intersection point are linearly dependent, while in case of transversal intersection, the normals of the intersecting surfaces at the intersection point are linearly independent.

Keywords: Parametric surface, Implicit surface, transversal intersection, tangential intersection.

Subject classification: 53A04, 53A05

1. Introduction

The intersection problems of surfaces is a fundamental process needed in modelling complex shapes in CAD/CAM system. Intersections are useful in the representation of the design of complex objects, in computer animations, and in NC machining for trimming off the region bounded by self intersection curves of offset surfaces [14]. For that reason the two types of surfaces commonly used in geometric modelling systems are parametric and implicit surfaces that lead to three types of surface-surface intersection (SSI) problems: parametric-parametric, implicit-implicit, implicit-parametric. The SSI is called transversal or tangential if the normal vectors of the surfaces are linearly independent or linearly dependent, respectively at the intersection point. In transversal intersection problems, the tangent vectors of the intersection curve can be found easily by the vector product of the normal vectors of the surfaces, while as in tangential intersection case such derivation is not possible.
The geometric properties of the parametrically defined curves can be found in the classical literature on differential geometry in [10, 12] and in the contemporary literature of geometric modelling [4, 6]. Also the higher curvatures of curves in $\mathbb{R}^n$ can be found in textbook [13] and in paper [5]. On the other hand, for the differential geometry of the intersection curves,

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there exists little literature in which transversal intersection is mostly studied. Willmore [12] obtained the unit tangent, the unit principal normal and unit binormal, as well as the curvature and the torsion of the intersection curve of two implicit surfaces in $\mathbb{R}^3$. Ye and Maekawa [14] gave algorithms for obtaining the Frenet apparatus of the intersection curves of two parametric surfaces in $\mathbb{R}^3$. They also derived algorithms for the evaluation of higher order derivatives for transversal as well as tangential intersections. Then Hartmann [2] gave formulae for computing the curvature and geodesic curvature of the intersection curves of all the three types of intersection problems in $\mathbb{R}^3$ using the implicit function theorem. Similarly Abdel-All et al.[8] provided an algorithm for the evaluation of the Frenet apparatus of the intersection curves of two implicit surfaces using implicit function theorem. Soliman et al.[7] obtained an algorithm for the Frenet apparatus of the intersection curves of two surfaces (implicit-parametric) in $\mathbb{R}^3$. Goldmann [11] derived closed formulae for computing the curvature and the torsion of the intersection curve of two implicit surfaces in $\mathbb{R}^3$ and the curvature of the intersection curves in $\mathbb{R}^{n+1}$. In [1] B.U Düldül and M. Çalişkan obtained the geodesic torsion of tangential intersection of two surfaces in $\mathbb{R}^3$, while as looking at an important characteristic(geodesic), we tried to find the geodesic curvature of tangential intersection of two regular parametric and implicit surfaces in $\mathbb{R}^3$.

2. Preliminaries

Definition 2.1. Let $\{e_1, e_2, e_3\}$ be the standard basis of three dimensional Euclidean space $\mathbb{R}^3$. The vector product of vectors $x = \sum_{i=1}^{3} x_i e_i$, $y = \sum_{i=1}^{3} y_i e_i$ is defined as

$$x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}. \tag{1}$$

The vector product yields a vector that is orthogonal to x and y.

Definition 2.2. let $M \subset E^3$ be a regular surface given by $R = R(u_1, u_2)$ and $\alpha : I \subset \mathbb{R} \to M$ be an arbitrary curve with arc length parametrisation. If $\{t, n, b\}$ is the moving Frenet Frame along α, then the Frenet formulas are given

$$\left.\begin{array}{rcl} t' & = & kn \\ n' & = & -kt + \tau b \\ b' & = & -\tau n \end{array}\right\} \tag{2}$$

where the factor k is called curvature and τ is called torsion. The torsion measures the rotation of the Frame about the tangent vector. The first two derivatives of the curve α are given by $\alpha' = t$ and $\alpha'' = t' = kn$. Also, since M is regular, the partial derivatives R_1, R_2 are linearly independent at every point of M, i.e., $R_1 \times R_2 \neq 0$, where $R_i = \frac{\partial R}{\partial u_i}$.
On the other hand since the curve $\alpha(s)$ lies on M, we can write $\alpha(s) = R(u_1(s), u_2(s))$, then we

have

$$\begin{aligned}
\alpha'(s) &= R_1 u_1' + R_2 u_2'. && (3)\\
\alpha''(s) &= R_{11}(u_1')^2 + 2R_{12}u_1'u_2' + R_{22}(u_2')^2 + R_1 u_1'' + R_2 u_2''. && (4)\\
\alpha'''(s) &= R_{111}(u_1')^3 + 3R_{112}(u_1')^2 u_2' + 3R_{122}u_1'(u_2')^2 + R_{222}(u_2')^3 + 3[R_{11}u_1'u_1'' \\
&\quad + R_{12}(u_1''u_2' + u_1'u_2'') + R_{22}u_2'u_2''] + R_1 u_1''' + R_2 u_2'''. && (5)
\end{aligned}$$

Definition 2.3. A unit surface normal vector of the parametric surface is defined as

$$\mathbf{N} = \frac{R_1 \times R_2}{\|R_1 \times R_2\|}.$$

Also, the normal curvature is obtained by projecting (4) onto $\mathbf{N}$, i.e.,

$$\kappa_n = L(u_1')^2 + 2Mu_1'u_2' + N(u_2')^2 \tag{6}$$

where L, M, N are the second fundamental form coefficients.

Definition 2.4. The geodesic curvature of $\alpha(s)$ at a point p is given by [3]

$$\begin{aligned}
k_g &= \left[\Gamma_{11}^2\left(\frac{du_1}{ds}\right)^3 + \left(2\Gamma_{12}^2 - \Gamma_{11}^1\right)\left(\frac{du_1}{ds}\right)^2\frac{du_2}{ds} + \left(\Gamma_{22}^2 - 2\Gamma_{12}^1\right)\frac{du_1}{ds}\left(\frac{du_2}{ds}\right)^2\right.\\
&\quad \left. -\Gamma_{22}^1\left(\frac{du_2}{ds}\right)^3 + \frac{du_1}{ds}\frac{d^2u_2}{ds^2} - \frac{d^2u_1}{ds^2}\frac{du_2}{ds}\right]\sqrt{EG-F^2}, && (7)
\end{aligned}$$

where $\Gamma_{jk}^i, (i,j,k = 1,2)$ are Christoffel symbols defined as follows:

$$\left.\begin{array}{ll}
\Gamma_{11}^1 = \frac{GE_{u_1} - 2FF_{u_1} + FE_{u_2}}{2(EG-F^2)}, & \Gamma_{11}^2 = \frac{2EF_{u_1} - EE_{u_2} - FE_{u_1}}{2(EG-F^2)},\\
\Gamma_{12}^1 = \frac{GE_{u_2} - FG_{u_1}}{2(EG-F^2)}, & \Gamma_{12}^2 = \frac{EG_{u_1} - FE_{u_2}}{2(EG-F^2)},\\
\Gamma_{22}^1 = \frac{2GF_{u_2} - GG_{u_1} - FG_{u_2}}{2(EG-F^2)}, & \Gamma_{22}^2 = \frac{EG_{u_2} - 2FF_{u_2} + FG_{u_1}}{2(EG-F^2)},
\end{array}\right\} \tag{8}$$

and E, F and G are the first fundamental coefficients.

3. Algorithm for geodesic Curvature of tangential intersection curve of two parametric surfaces.

Let M_1 and M_2 be two regular intersecting surfaces given by parametric equations $R^A = R^A(u_1,u_2)$ and $R^B = R^B(v_1,v_2)$, respectively, then the unit normal vector of these surfaces are

$$\mathbf{N}^i = \frac{R_1^i \times R_2^i}{R_1^i \times R_2^i}, \quad i = A,B. \tag{9}$$

We assume that the intersection of these surfaces is a smooth curve $\alpha(s)$ with arc length parametrisation and $\alpha(s_0) = p$ be a point on the intersection curve. Now to find the geodesic curvature of the tangential intersection curve with respect to R^A, from (7) we need to find the unknowns u_i' and u_i'', $i=1,2$. Since the intersection is tangential, the tangent vector of the intersection curve is given by

$$t = R_1^A u_1' + R_2^A u_2' = R_2^B v_1' + R_2^B v_2'. \tag{10}$$

Suppose $\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}$(say), from $k_n = K.\mathbf{N}$, we get

$$L^A(u_1')^2 + 2M^A(u_1'u_2') + N^A(u_2')^2 = L^B(v_1')^2 + 2M^B(v_1'v_2') + N^B(v_2')^2, \tag{11}$$

where (L^A, M^A, N^A) and (L^B, M^B, N^B) are the second fundamental coefficients of R^A and R^B, respectively.

Equations (10) and (11) form a system of four nonlinear equations in four unknowns (u_1', u_2', v_1', v_2'). This nonlinear system can be solved by representing v_1' and v_2' in terms of linear combinations of u_1' and u_2' from (10) and then substituting the results in (11). Taking the cross product of both sides of (10) with R_1^B and R_2^B, and projecting the resulting equations onto the common surface normal $\mathbf{N}$ at p, we have

$$v_1' = \delta_{11}u_1' + \delta_{12}u_2', \tag{12}$$

$$v_2' = \delta_{21}u_1' + \delta_{22}u_2', \tag{13}$$

where

$$\delta_{11} = \frac{(R_1^A \times R_2^B)\cdot\mathbf{N}}{(R_1^B \times R_2^B)\cdot\mathbf{N}} = \frac{det(R_1^A, R_2^B, \mathbf{N})}{\sqrt{E^B G^B - (F^B)^2}}, \tag{14}$$

$$\delta_{12} = \frac{(R_2^A \times R_2^B)\cdot\mathbf{N}}{(R_1^B \times R_2^B)\cdot\mathbf{N}} = \frac{det(R_2^A, R_2^B, \mathbf{N})}{\sqrt{E^B G^B - (F^B)^2}}, \tag{15}$$

$$\delta_{21} = \frac{(R_1^B \times R_1^A)\cdot\mathbf{N}}{(R_1^B \times R_2^B)\cdot\mathbf{N}} = \frac{det(R_1^B, R_1^A, \mathbf{N})}{\sqrt{E^B G^B - (F^B)^2}}, \tag{16}$$

$$\delta_{22} = \frac{(R_1^B \times R_2^A)\cdot\mathbf{N}}{(R_1^B \times R_2^B)\cdot\mathbf{N}} = \frac{det(R_1^B, R_2^A, \mathbf{N})}{\sqrt{E^B G^B - (F^B)^2}}. \tag{17}$$

Substituting (12) and (13) in (11), we obtain

$$\gamma_{11}(u_1')^2 + 2\gamma_{12}(u_1')(u_2') + \gamma_{22}(u_2')^2 = 0, \tag{18}$$

where

$$\begin{aligned}
\gamma_{11} &= \delta_{11}^2 L^B + 2\delta_{11}\delta_{21}M^B + \delta_{21}^2 N^B - L^A,\\
\gamma_{12} &= \delta_{11}\delta_{12}L^B + 2(\delta_{11}\delta_{22} + \delta_{21}\delta_{12})M^B + \delta_{21}\delta_{22}N^B - M^A,\\
\gamma_{22} &= \delta_{12}^2 L^B + 2\delta_{12}\delta_{22}M^B + \delta_{22}^2 N^B - N^A.
\end{aligned}$$

Denoting $\rho = \frac{u_1'}{u_2'}$ when $c_{11} \neq 0$ or $\mu = \frac{u_2'}{u_1'}$ when $b_{11} = 0$ and $b_{22} \neq 0$ and solving (18) for ρ or μ, then t can be computed by means of either of the following

$$t = \frac{\rho R_1^A + R_2^A}{|\rho R_1^A + R_2^A|}, \ \ or \ \ t = \frac{R_1^A + \mu R_2^A}{|R_1^A + \mu R_2^A|}. \tag{19}$$

Now since t is known from (19), taking the dot product of (10) with R_1^A and R_2^A, we get the unknowns u_i', $i = 1,2$ by solving the following system of equations

$$(R_1^A \cdot R_1^A)u_1' + (R_1^A \cdot R_2^A)u_2' = R_1^A \cdot t,$$
$$(R_1^A \cdot R_2^A)u_1' + (R_2^A \cdot R_2^A)u_2' = R_2^A \cdot t.$$

Remark 3.1. The number of real intersection points of the surfaces arises four distinct cases of (18).

1.If $\gamma_{11}^2 - \gamma_{11}\gamma_{22} < 0$, then (18) does not have any solution, hence the surfaces do not intersect each other implying p is an isolated contact point of R^A and R^B.

2.If $\gamma_{11}^2 - \gamma_{11}\gamma_{22} = 0$ and $\gamma_{11}^2 + \gamma_{12}^2 + \gamma_{22}^2 \neq 0$, then (18) has double root implying R^A and R^B have one point in common, hence the tangent direction is unique.

3. If $\gamma_{11}^2 - \gamma_{11}\gamma_{22} > 0$, then (18) has distinct roots. Then p is a branch point of the intersection curve $\alpha(s)$.

4. If $\gamma_{11} = \gamma_{12} = \gamma_{22} = 0$, then (18) vanishes for any value of u_1' and u_2'. Thus R^A and R^B overlap and so has a contact of higher order at p.

To find the geodesic curvature of the tangential intersection curve with respect to surface R^A, we still need to find the second derivatives u_i'', $i = 1,2$. The curvature vector of the intersection curve $\alpha(s)$ at point p can be expressed as

$$\left.\begin{aligned} \alpha''(s) &= R_{11}^A(u_1')^2 + 2R_{12}^A u_1'u_2' + R_{22}^A(u_2')^2 + R_1^A u_1'' + R_2^A u_2'' \\ &= R_{11}^B(v_1')^2 + 2R_{12}^B v_1'v_2' + R_{22}^B(v_2')^2 + R_1^B v_1'' + R_2^B v_2''. \end{aligned}\right\} \tag{20}$$

In (20), we have four unknowns $(u_1'', u_2'', v_1'', v_2'')$, thus to obtain them we need four equations. Rewrite (20) as

$$R_1^A u_1'' + R_2^A u_1'' = R_1^B v_1'' + R_2^B v_2'' + \Omega, \tag{21}$$

where

$$\Omega = R_{11}^B(v_1')^2 + 2R_{12}^B v_1'v_2' + R_{22}^B(v_2')^2 - R_{11}^A(u_1\prime)^2 - 2R_{12}^A(u_1')(u_2') - R_{22}^A(u_2')^2. \tag{22}$$

Taking the cross product of (21) with R_2^B and R_1^B and projecting onto the unit surface normal, we get

$$v_1'' = \delta_{11}u_1'' + \delta_{12}u_2'' + \delta_{13}, \tag{23}$$

$$v_2'' = \delta_{21}u_1'' + \delta_{22}v_1'' + \delta_{23}, \tag{24}$$

where $\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}$ are coefficients defined in (14) – (17), and δ_{13}, δ_{23} are the coefficients defined as follows

$$\delta_{13} = \frac{(R_2^B \times \Omega)\cdot \mathbf{N}}{(R_1^B \times R_2^B)\cdot \mathbf{N}} = \frac{det(R_2^B, \Omega, \mathbf{N})}{\sqrt{E^B G^B - (F^B)^2}}, \tag{25}$$

$$\delta_{23} = \frac{(\Omega \times R_1^B)\cdot \mathbf{N}}{(R_1^B \times R_2^B)\cdot \mathbf{N}} = \frac{det(R_1^B, \Omega, \mathbf{N})}{\sqrt{E^B G^B - (F^B)^2}}. \tag{26}$$

Since the curvature vector k is orthogonal to tangent vector t, therefore we have

$$\alpha'' \cdot t = q_1 u_1'' + q_2 u_2'' + q_3 = 0, \tag{27}$$

where $q_1 = (R_1^A \cdot t)$, $q_2 = (R_1^A \cdot t)$, $q_3 = (R_{11}^A \cdot t)(u_1')^2 + 2(R_{12}^A \cdot t)u_1' u_2' + (R_{22}^A \cdot t)(u_2')^2$.
To find (u_1'', u_2'') the last equation can be found by differentiating $\alpha(s) = R^A(u_1(s), u_2(s)) = R^B(v_1(s), v_2(s))$ three times and projecting onto the common normal vector $\mathbf{N}$. Thus from (5), we get

$$\begin{aligned} 3[L^A u_1' u_1'' + M^A(u_1'' u_2' + u_1' u_2'') + N^A u_2' u_2''] + III^A = \\ 3[L^B v_1' v_1'' + M^B(v_1'' v_2' + v_1' v_2'') + N^B v_2' v_2''] + III^B, \end{aligned} \tag{28}$$

where

$$III^A = R_{111}^A \cdot N^A (u_1')^3 + 3R_{112}^A \cdot N^A (u_1')^2 u_2' + 3R_{122}^A \cdot N^A u_1' (u_2')^2 + R_{222}^A \cdot N^A (u_2')^3,$$

and

$$III^B = R_{111}^B \cdot N^B (v_1')^3 + 3R_{112}^B \cdot N^B (v_1')^2 v_2' + 3R_{122}^B \cdot N^B v_1' (v_2')^2 + R_{222}^B \cdot N^B (v_2')^3.$$

Example 3.1. *Consider the two surfaces R^A and R^B given by parametric equations*

$$R^A = (\cos u_1 - \cos u_1 \cos u_2 + \sin u_1 \sin u_2, 3\sin u_1 - \sin u_1 \cos u_2 - \cos u_1 \sin u_2, \sin u_2); \;\; 0 \le u_1, u_2 \le 2\pi,$$

$$R^B = (2\cos v_1, 2\sin v_1, v_2); \;\; 0 \le v_1, v_2 \le 2\pi.$$

Let $p = R^A(0,0) = R^B(0,0) = (2,0,0)$ be a point on the intersection curve of R^A and R^B.
The non-vanishing partial derivatives of the surface R^A are given by
$R_1^A = (0,2,0)$, $R_2^A = (0,-1,1)$, $R_{12}^A = (1,0,0)$, $R_{22}^A = (1,0,0)$,
$R_{111}^A = (0,-2,0)$, $R_{112}^A = (0,1,0)$, $R_{122}^A = (0,1,0)$, $R_{222}^A = (0,1,-1)$.
Hence the unit normal vector for R^A is obtained as

$$\boldsymbol{N}^A = (1,0,0). \tag{29}$$

The first and second fundamental coefficients of R^A are

$$E^A = 4, \quad F^A = -2, \quad G^A = 2, \quad L^A = 0, \quad M^A = N^A = 1.$$

Similarly the non-vanishing partial derivatives of the surface R^B are
$R_1^B = (0,2,0)$, $R_2^B = (0,0,1)$, $R_{11}^B = (-2,0,0)$, $R_{111}^B = (0,-2,0)$.
The first and second fundamental coefficients of the surface R^B are

$$E^B = 4, \quad G^B = 1, \quad F^B = 0, \quad L^B = -2, \quad M^B = N^B = 0.$$

Also the unit normal to the surface R^B is

$$\boldsymbol{N}^B = (1,0,0). \tag{30}$$

Therefore from (29) and (30) we see that the intersection is tangential at p.
Using the above algorithm, we obtain

$$u_1' = 0.698795, \quad u_2' = 0.806898, \quad v_1' = 0.806898, \quad v_2' = 0.295345.$$

From (19) the unit tangent of the intersection curve is given by

$$t = (0, 0.590691, 0.806898).$$

Using the above algorithm again, we easily obtain

$$u_1'' = u_2'' = 0.$$

Therefore from (8), we obtain

$$\Gamma_{11}^1 = \frac{-1}{4}, \Gamma_{11}^2 = \frac{-1}{4}, \Gamma_{12}^1 = \frac{1}{2}, \Gamma_{12}^2 = \frac{3}{4}, \Gamma_{22}^1 = \frac{1}{4}, \Gamma_{22}^2 = \frac{3}{4}.$$

Consequently, we obtain $\kappa_g^{R^A} = 0.833502$*(See figure 1). Similarly, we can find the geodesic curvature of the intersection curve with respect to* R^B.

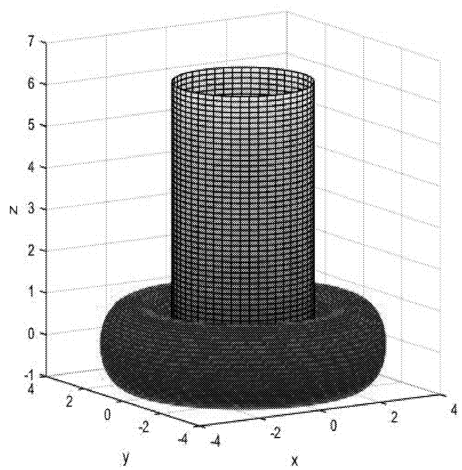

Figure 1:

Example 3.2. *Consider two parametric surfaces* R^A *and* R^B *given by*

$$R^A = (\sin u_2 \cos u_1, \sin u_1 \sin u_2, \cos u_2); \quad 0 \le u_1, u_2 \le 2\pi,$$

$$R^B = (\cos v_1 \cos v_2, \sin v_1 \cos v_2, -\sin v_2); \quad 0 \le v_1, v_2 \le 2\pi.$$

Then at point p, we have

$$p = R^A(0, \pi/2) = R^B(0, 0) = (1, 0, 0).$$

The unit normals are given by $\mathbf{N}^A = \mathbf{N}^B = (-1,0,0)$*, implying the intersection is tangential. Also from (18), we obtain* $\gamma_{11} = \gamma_{12} = \gamma_{22} = 0$.
Thus, the surfaces have at least second order contact as it is evident by the overlapping of two surfaces in figure 2 below.

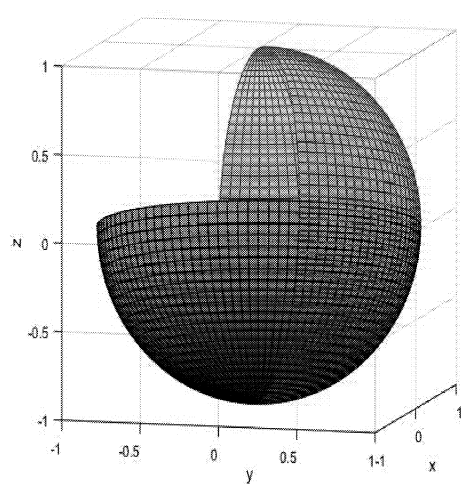

Figure 2:

4. Algorithm for geodesic curvature of tangential intersection of two implicit surfaces

Let $R^A = R^A(x_1,x_2,x_3) = 0$ and $R^B = R^B(x_1,x_2,x_3) = 0$ be two implicit surfaces, then the geodesic curvature of the tangential intersection curve with respect to surface R^A is given by [9]

$$\kappa_g^{R^A} = \frac{1}{||R^A||}(x_2'x_3'' - x_2''x_3')R_1^A + (x_3'x_1'' - x_3''x_1')R_2^A + (x_1'x_2'' - x_1''x_2')R_3^A. \tag{31}$$

Thus, to find the the geodesic curvature of the intersection curve with respect to R^A, we have to find (x_1',x_2',x_3') and (x_1'',x_2'',x_3''). For that, we find the tangent vector and curvature vector.

Definition 4.1. The unit normal vector of implicit surfaces R^A and R^B are defined as

$$\mathbf{N}^A = \frac{\nabla R^A}{||\nabla R^A||}, \quad \mathbf{N}^B = \frac{\nabla R^B}{||\nabla R^B||},$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$. Also, since the intersection is tangential, it follows that $\mathbf{N}^A || \mathbf{N}^B$.

Consider $\alpha(x_1)$ be a curve parameterised by x_1 and formed by the intersection of R^A and R^B at the tangential point p. Then by choosing the orientation of surfaces suitably, we have

$$\frac{\nabla R^A}{||\nabla R^A||} = \frac{\nabla R^B}{||\nabla R^B||}, \tag{32}$$

Let μ be the scalar function defined as $\mu = \frac{\|\nabla R^A\|}{\|\nabla R^B\|}$, then (32) can be written as

$$\nabla R^A = \mu \nabla R^B. \tag{33}$$

4.1. Tangent vector for the tangential intersection of two implicit surfaces.

The intersection curve $\alpha(x_1)$ of the implicit surfaces is given by

$$\alpha(x_1) = \{\alpha(x_1, x_2(x_1), x_3(x_1)) | R^A = 0 \cap R^B = 0\}.$$

Now for R^A and R^B, we have

$$\nabla R^A \cdot \dot{\alpha} = 0, \quad \nabla R^B \cdot \dot{\alpha} = 0, \quad \nabla R^A \cdot \ddot{\alpha} = -\dot{\alpha}^T H_{R^A} \dot{\alpha} \quad \text{and} \quad \nabla R^B \cdot \ddot{\alpha} = -\dot{\alpha}^T H_{R^B} \dot{\alpha} \tag{34}$$

where $\nabla R^A(\alpha(x_1)) = [R_1^A, \quad R_2^A, R_3^A]^T, \quad \dot{\alpha} = [1, \dot{x_2}, \quad \dot{x_3}]^T, \ddot{\alpha} = [0, \ddot{x_2}, \quad \ddot{x_3}]^T, \quad \dddot{\alpha} = [0, \dddot{x_2}, \quad \dddot{x_3}]^T$ and H_{R^A} and H_{R^B} are the Hessian matrices of R^A and R^B given by

$$H_{R^A} = \begin{bmatrix} R_{11}^A & R_{12}^A & R_{13}^A \\ R_{12}^A & R_{22}^A & R_{23}^A \\ R_{13}^A & R_{23}^A & R_{33}^A \end{bmatrix} \text{ and } \quad H_{R^B} = \begin{bmatrix} R_{11}^B & R_{12}^B & R_{13}^B \\ R_{12}^B & R_{22}^B & R_{23}^B \\ R_{13}^B & R_{23}^B & R_{33}^B \end{bmatrix}.$$

Indeed for $R^A \cdot \dot{\alpha} = 0$, we have

$$\begin{aligned} &\frac{d}{dx_1}(R^A(\alpha(x_1))) = 0, \\ &R_1^A x_1 + R_2^A x_2 + R_3^A x_3 = 0, \\ &\nabla R^A \cdot \dot{\alpha} = 0 \end{aligned}$$

and for $\nabla R^A \cdot \ddot{\alpha} = -\dot{\alpha}^T H_{R^A} \dot{\alpha}$, we have

$$\begin{aligned} &\frac{d}{dx_1}\left(\frac{d}{dx_1} R^A(\alpha(x_1))\right) = 0, \\ &(R_{11}^A \dot{x}_1 + R_{12}^A \dot{x}_2 + R_{13}^A \dot{x}_3)\dot{x}_1 + R_1^A \ddot{x}_1 \\ &+(R_{12}^A \dot{x}_1 + R_{22}^A \dot{x}_2 + R_{23}^A \dot{x}_3)\dot{x}_2 + R_2^A \ddot{x}_2 \\ &+(R_{13}^A \dot{x}_1 + R_{23}^A \dot{x}_2 + R_{33}^A \dot{x}_3)\dot{x}_3 + R_3^A \ddot{x}_3 = 0, \\ &\nabla R^A \cdot \ddot{\alpha}(x_1) + (\dot{\alpha}(x_1))^T H_{R^A} \dot{\alpha}(x_1) = 0. \end{aligned}$$

Projecting $\ddot{\alpha}(x_1)$ onto the normal vector of R^A and using (33), we obtain

$$\langle \ddot{\alpha}(x_1), \nabla R^A \rangle = \mu \langle \ddot{\alpha}(x_1), \nabla R^B \rangle. \tag{35}$$

Using (34), we get

$$\dot{\alpha}^T (H_{R^A} - \mu H_{R^B}) \dot{\alpha} = 0, \tag{36}$$

or

$$\begin{aligned} &(R_{22}^A - \mu R_{22}^B)(\dot{x_2})^2 + (R_{33}^A - \mu R_{33}^B)(\dot{x_3})^2 + 2(R_{23}^A - \mu R_{23}^B)\dot{x_2}\dot{x_3} \\ &+2(R_{12}^A - \mu R_{12}^B)\dot{x_2} + 2(R_{13}^A - \mu R_{13}^B)\dot{x_3} + (R_{11}^A - \mu R_{11}^B) = 0. \end{aligned} \tag{37}$$

Since $\nabla R^A \cdot \dot{\alpha} = 0$, we have

$$R_1^A + R_2^A \dot{x_2} + R_3^A \dot{x_3} = 0, \tag{38}$$

which gives

$$\dot{x_3} = \frac{R_1^A + R_2^A \dot{x_2}}{R_3^A}, \quad R_3^A \neq 0. \tag{39}$$

Using (39) in (38), we get

$$d_{11}(\dot{x_2})^2 + 2d_{12}\dot{x_2} + d_{13} = 0, \tag{40}$$

where

$$\begin{aligned}
d_{11} &= (R_{22}^A - \mu R_{22}^B)(R_3^A)^2 + (R_{33}^A - \mu R_{33}^B)(R_2^A)^2 - 2(R_{23}^A - R_{23}^B)R_2^A R_3^A, \\
d_{12} &= (R_{33}^A - \mu R_{33}^B)R_1^A R_2^A R_3^A - (R_{23}^A - \mu R_{23}^B)R_1^A R_3^A - (R_{12}^A - \mu R_{12}^B)(R_3^A)^2 \\
&\quad -(R_{13}^A - \mu R_{13}^B)R_2^A R_3^A, \\
d_{13} &= (R_{33}^A - \mu R_{33}^B)(R_1^A)^2 - 2(R_{13}^A - \mu R_{13}^B)R_1^A R_3^A + (R_{11}^A - \mu R_{11}^B)(R_3^A)^2.
\end{aligned}$$

Thus (40) gives

$$\dot{x_2} = \frac{-d_{12} \pm \sqrt{(d_{12})^2 - d_{11}d_{13}}}{d_{11}}, d_{11} \neq 0. \tag{41}$$

Hence from (41) and (39), $\dot{\alpha}(x_1)$ is obtained, consequently the unit tangent vector is given by $t = \frac{\dot{\alpha}(x_1)}{\|\dot{\alpha}(x_1)\|}$. Moreover, remark 1 also follows here.

4.2. Curvature vector for the tangential intersection of two implicit surfaces.

In this subsection, we find the second order derivatives $\ddot{x}_i$, for that first we have the following lemma.

Lemma 4.1. *For an implicit surface $R = R(x_1, x_2, x_3)$ and $\alpha(x_1)$ be a curve on R, then we have*

$$\nabla R \cdot \dddot{\alpha}(x_1) = -(\dot{\alpha})^T \frac{d}{dx_1}(H_R)\dot{\alpha} - 3(\dot{\alpha})^T H_R \ddot{\alpha} \tag{42}$$

where

$$\frac{d}{dx_1}(H_R) = \begin{bmatrix} \frac{\partial}{\partial x_1}(H_R) & \frac{\partial}{\partial x_2}(H_R) & \frac{\partial}{\partial x_3}(H_R) \end{bmatrix}^T \cdot \alpha(\dot{x}_1),$$

$$\tfrac{\partial H_R}{\partial x_1} = \begin{bmatrix} R_{111} & R_{121} & R_{131} \\ R_{211} & R_{221} & R_{231} \\ R_{311} & R_{321} & R_{331} \end{bmatrix}, \tfrac{\partial H_R}{\partial x_2} = \begin{bmatrix} R_{112} & R_{122} & R_{132} \\ R_{212} & R_{222} & R_{232} \\ R_{312} & R_{322} & R_{332} \end{bmatrix}, \tfrac{\partial H_R}{\partial x_3} = \begin{bmatrix} R_{113} & R_{123} & R_{133} \\ R_{213} & R_{223} & R_{233} \\ R_{313} & R_{323} & R_{333} \end{bmatrix}$$

.

Proof. Since, we have

$$\frac{d}{dx_1}\left(\frac{d^2}{dx_1^2}R(\alpha(x_1))\right) = 0,$$

or

$$\left\{\begin{array}{l} \{(R_{111}\dot{x}_1+R_{112}\dot{x}_2+R_{113}\dot{x}_3)\dot{x}_1+R_{11}\ddot{x}_1+(R_{121}\dot{x}_1+R_{122}\dot{x}_2+R_{123}\dot{x}_3)\dot{x}_2+R_{12}\ddot{x}_2 \\ +(R_{131}\dot{x}_1+R_{132}\dot{x}_2+R_{133}\dot{x}_3)\dot{x}_3+R_{13}\ddot{x}_2\}\dot{x}_1+2(R_{11}\dot{x}_1+R_{12}\dot{x}_2+R_{13}\dot{x}_3)\ddot{x}_1+R_1\dddot{x}_1 \\ \{(R_{112}\dot{x}_1+R_{122}\dot{x}_2+R_{123}\dot{x}_3)\dot{x}_1+R_{12}\ddot{x}_1+(R_{122}\dot{x}_1+R_{222}\dot{x}_2+R_{223}\dot{x}_3)\dot{x}_2+R_{22}\ddot{x}_2 \\ +(R_{123}\dot{x}_1+R_{223}\dot{x}_2+R_{233}\dot{x}_3)\dot{x}_3+R_{23}\ddot{x}_2\}\dot{x}_2+2(R_{12}\dot{x}_1+R_{22}\dot{x}_2+R_{23}\dot{x}_3)\ddot{x}_2+R_2\dddot{x}_3 \\ \{(R_{113}\dot{x}_1+R_{123}\dot{x}_2+R_{133}\dot{x}_3)\dot{x}_1+R_{13}\ddot{x}_1+(R_{123}\dot{x}_1+R_{223}\dot{x}_2+R_{233}\dot{x}_3)\dot{x}_2+R_{23}\ddot{x}_2 \\ +(R_{133}\dot{x}_1+R_{233}\dot{x}_2+R_{333}\dot{x}_3)\dot{x}_3+R_{33}\ddot{x}_2\}\dot{x}_3+2(R_{13}\dot{x}_1+R_{23}\dot{x}_2+R_{33}\dot{x}_3)\ddot{x}_3+R_3\dddot{x}_3 \end{array}\right\}=0,$$

or

$$\left\{\begin{array}{l} ((\dot{\alpha})^T\frac{\partial}{\partial x_1}(H_R)\dot{\alpha}+\nabla R_1\cdot\ddot{\alpha})\dot{x}_1+2(\nabla R_1\cdot\dot{\alpha})\ddot{x}_1+R_1\dddot{x}_1 \\ +((\dot{\alpha})^T\frac{\partial}{\partial x_2}(H_R)\dot{\alpha}+\nabla R_2\cdot\ddot{\alpha})\dot{x}_2+2(\nabla R_2\cdot\dot{\alpha})\ddot{x}_2+R_2\dddot{x}_2 \\ +((\dot{\alpha})^T\frac{\partial}{\partial x_3}(H_R)\dot{\alpha}+\nabla R_3\cdot\ddot{\alpha})\dot{x}_3+2(\nabla R_3\cdot\dot{\alpha})\ddot{x}_3+R_3\dddot{x}_3 \end{array}\right\}=0,$$

The above can be reduced to

$$\nabla R\cdot\dddot{\alpha}+(\dot{\alpha})^T\frac{d}{dx_1}(H_R)\dot{\alpha}+3(\dot{\alpha})^T H_R\ddot{\alpha}=0.$$

which proves the lemma. □

Now in order to compute $\dddot{x}_i$, projecting $\dddot{\alpha}(x_1)$ onto normal vector and using (33), we get

$$\langle\dddot{\alpha},\nabla R^A\rangle=\mu\langle\dddot{\alpha},\nabla R^B\rangle. \tag{43}$$

Using (42), we obtain

$$-3(\dot{\alpha})^T H_A\ddot{\alpha}-(\dot{\alpha})^T((\nabla H_A)\dot{\alpha})\dot{\alpha}=\mu(-3(\dot{\alpha})^T H_A\ddot{\alpha}-(\dot{\alpha})^T((\nabla H_B)\dot{\alpha})\dot{\alpha}),$$

where $(\nabla H_i)\dot{\alpha}=\frac{d}{dx_1}(H_i(\alpha(x_1)))$,
or

$$3(\dot{\alpha})^T(H_{R^A}-\mu H_{R^B})\ddot{\alpha}=(\dot{\alpha})^T((\mu\nabla H_{R^B}-\nabla H_{R^A})\dot{\alpha})\dot{\alpha}. \tag{44}$$

Now (44) and the last equation of (34) can be written as

$$\begin{bmatrix} 3(\dot{\alpha})^T(H_{R^A}-\mu H_{R^B}) \\ \nabla R^B \end{bmatrix}\begin{bmatrix} \dddot{x}_2 \\ \dddot{x}_3 \end{bmatrix}=\begin{bmatrix} (\dot{\alpha})^T((\mu\nabla H_{R^B}-\nabla H_{R^A})\dot{\alpha})\dot{\alpha} \\ -(\dot{\alpha})^T H_{R^B}\dot{\alpha} \end{bmatrix},$$

or

$$\begin{bmatrix} \dddot{x}_2 \\ \dddot{x}_3 \end{bmatrix}=\varphi^{-1}\begin{bmatrix} (\dot{\alpha})^T((\mu\nabla H_{R^B}-\nabla H_{R^A})\dot{\alpha})\dot{\alpha} \\ -(\dot{\alpha})^T H_{R^B}\dot{\alpha} \end{bmatrix},\varphi^{-1}\neq 0, \tag{45}$$

where

$$\varphi=\begin{bmatrix} 3(\dot{\alpha})^T(H_{R^A}-\mu H_{R^B}) \\ \nabla R^B \end{bmatrix}.$$

Equation (45) gives the desired derivative.

Example 4.1. *Consider two implicit surfaces given by*

$$R^A = \frac{100}{9}x_1^2 + \frac{25}{36}x_2^2 + \frac{25}{16}x_3^2 = 7$$

and

$$R^B = \frac{25}{9}x_1^2 + x_2^2 + \frac{25}{16}x_3^2 = 7.$$

The normal vectors for R^A and R^B are given by
$\nabla R^A = (\frac{200}{9}x_1, \frac{25}{18}x_2, \frac{25}{8}x_3)$, $\nabla R^B = (\frac{50}{9}x_1, 2x_2, \frac{25}{8}x_3)$, *respectively.*
For the point of intersection $p = (0,0,0.8) \in R^A \cap R^B$, we have

$$\nabla R^A = (0,0,1) \text{ and } \nabla R^B = (0,0,1).$$

Thus the two surfaces intersects tangentially at p (see figure 3) and from (31) $\mu = 1$.
The non-vanishing first and second order coefficients for R^A and R^B are obtained as

$$R_3^A = R_3^B = \frac{5}{2}, \quad R_{11}^A = \frac{200}{9}, \quad R_{22}^A = \frac{25}{18}, \quad R_{33}^A = \frac{25}{8}, \quad R_{11}^B = \frac{50}{9}, \quad R_{22}^B = 2, \quad R_{33}^B = \frac{25}{8}.$$

Hence, we obtain

$$d_{11} = -\frac{272}{72}, \quad d_{12} = 0, \quad d_{13} = \frac{625}{6},$$

which implies that p is a branch point.
Thus from (39), (41), we obtain $\dot{\alpha} = \left(1, \pm\frac{25}{2}\sqrt{\frac{3}{17}}, 0\right)$ and from (45), we obtain $\ddot{\alpha} = \left(0, 0, \frac{3715}{153}\right)$.
Consequently from (31), we obtain $k_g^{R^A} = 0$. Similarly, we can find $k_g^{R^B}$.

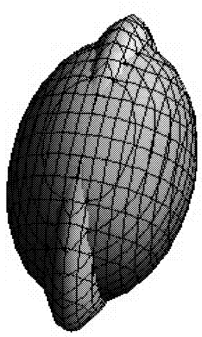

Figure 3:

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